PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 9 JANUARY 15, 1923 Number 1

ON RIEMANN SPACES CONFORMAL TO EUCLIDEAN SPACE

By H. W. Brinkmann

Department of Mathematics, Harvard University

Communicated, November 22, 1922

The purpose of this note is to call attention to the following theorem which appears to have escaped notice.

Theorem. An n-dimensional Riemann space conformal to Euclidean space can be considered as the intersection of a null hypersphere and another hypersurface in $(n + 2)$ − dimensional Euclidean space. Conversely, the intersection of a null hypersphere and a hypersurface in $(n + 2)$ − dimensional Euclidean space gives an n-dimensional Riemann space which is conformal to Euclidean space.

In particular, then, an n-dimensional Riemann space conformal to Euclidean space can always be "imbedded" in $(n + 2)$ − dimensional Euclidean space

The proof is very simple. Consider the transformation.

$$y_i = x_i\, \varphi(x_1, x_2, \ldots x_n),\ i = 1, 2, \ldots n, \qquad (1)$$

where φ is any function of $x_1, x_2, \ldots, x_n$ having continuous first partial derivatives which does not vanish at the origin. The Jacobian of the transformation will, under these conditions, not vanish at the origin. Then

$$dy_i = \varphi dx_i + x_i d\varphi,$$

and so

$$dy_1^2 + dy_2^2 + \ldots + dy_n^2 = \varphi^2(dx_1^2 + \ldots + dx_n^2) + 2r\varphi\, dr\, d\varphi + r^2\, d\varphi^2,$$

where

$$r^2 = x_1^2 + x_2^2 + \ldots + x_n^2.$$

If we put

$$x_{n+1} = \frac{1}{2}\left(\varphi r^2 - \frac{1}{\varphi}\right), \; x_{n+2} = \frac{1}{2}\sqrt{-1}\left(\varphi r^2 + \frac{1}{\varphi}\right), \qquad (2)$$

the identity just given is readily seen to be equivalent to

$$\frac{1}{\varphi^2}(dy_1^2 + \ldots + dy_n^2) = dx_1^2 + \ldots + dx_n^2 + dx_{n+1}^2 + dx_{n+2}^2. \qquad (3)$$

Due to the transformation (1), φ can be considered a function either of $x_1, \ldots, x_n$, or of $y_1, \ldots, y_n$. So it appears, by (3), that any Riemann space conformal to Euclidean space can be considered as an n-dimensional "surface" in $(n + 2)$ − dimensional Euclidean space. The equations of this "surface" are precisely (2). We readily verify that

$$x_1^2 + x_2^2 + \ldots + x_n^2 + x_{n+1}^2 + x_{n+2}^2 = 0, \qquad (4)$$

hence our "surface" is the intersection of the null hypersphere (4) and either one of the hypersurfaces (2). This proves the direct theorem.

Conversely, let an n-dimensional "surface" be given by

$$x_{n+1} = \psi_1(x_1, \ldots, x_n), x_{n+2} = \psi_2(x_1, \ldots, x_n). \qquad (5)$$

If (4) is satisfied we can make (5) identical with (2) by taking

$$\varphi = -\frac{1}{\psi_1 + i\psi_2},$$

and, by (3), the converse is established.

By the aid of this theorem it is very easy to obtain Schouten's necessary and sufficient condition that a given Riemann space be conformal to Euclidean space.[1] If the line element of the Riemann space is given by

$$ds^2 = g_{\alpha\beta}\, dx_\alpha\, dx_\beta, \; |g_{ij}| \neq 0,$$

the generalized Gauss-Mainardi-Codazzi equations become, for us,

$$R_{ijkl} + (g_{jl} M_{ik} + g_{ik} M_{jl} - g_{jk} M_{il} - g_{il} M_{jk}) = 0 \qquad (6)$$

$$M_{ij/k} = M_{ik/j}, \qquad (7)$$

where R_{ijkl} is the well-known Riemann tensor, M_{ij} is a symmetric tensor and $M_{ij/k}$ its covariant derivative. Equations (6) and (7) are readily found to be necessary and sufficient that it be possible to consider the given Riemann space as the section of a null hypersphere with some other

hypersurface in Euclidean space of dimensionality $n + 2$.[2] Hence the Riemann space is conformal to Euclidean space if and only if a symmetric tensor M_{ij} satisfying (6) and (7) can be found. This is precisely Schouten's condition. Schouten has shown, in addition, that (7) is a consequence of (6) if $n > 3$.

The device used in the proof of the theorem given in this note can, with certain modifications, be applied to much more general situations. The results obtained in this manner will be given in a subsequent note.

[1] Schouten, J. A., *Math. Zeitschr.*, **11**, 1921 (58–88).

[2] For the equations used here see: Ricci, G., *Lezioni sulla Teoria delle Superficie, Padua* 1898, pp. 87 *et seq.*

EQUIAFFINE GEOMETRY OF PATHS

BY O. VEBLEN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Read before the Academy, November 16, 1922

By an equiaffine geometry we mean the group of theorems in an affine geometry which relate to a particular definition of volume but not necessarily of distance. Let us inquire as to the restriction to be imposed on the functions $\Gamma^i_{\alpha\beta}$ which determine an affine geometryof paths (these PROCEEDINGS, **8**, p. 347) in order that there shall exist a definition of volume which generalizes that used in the Riemann geometry. Volume in the Riemann geometry is defined by the invariant integral

$$\int \sqrt{g}\, dx^1\, dx^2 \ldots dx^n$$

where g is the determinant $|g_{ij}|$. It is known that

$$\Gamma^i_{i\alpha} = \partial \log \sqrt{g}/\partial x^\alpha.$$

This can be generalized to the case in which there is no distance tensor g_{ij} provided that the skew symmetric tensor

$$S_{\alpha\beta} = B^i_{i\alpha\beta} = \partial\Gamma^i_{i\alpha}/\partial x^\beta - \partial\Gamma^i_{i\beta}/\partial x^\alpha$$

vanishes. For the vanishing of this tensor is the necessary and sufficient condition that there exist a function $\gamma(x^1, x^2, \ldots, x^n)$ such that

$$\Gamma^i_{i\alpha} = \partial \log \gamma/\partial x^\alpha.$$

According to (3.3) loc. cit., if we transform from $(x^1, x^2, \ldots, x^n)$ to $(z^1, z^2, \ldots, z^n)$ the functions $\Gamma^i_{i\alpha}$ are transformed as follows:

$$\bar{\Gamma}^i_{i\alpha}(z) = \Gamma^i_{ij}(x)\frac{\partial x^j}{\partial z^\alpha} + \frac{\partial}{\partial z^\alpha}\left(\frac{\partial x^j}{\partial z^i}\right)\cdot\frac{\partial z^i}{\partial x^j}$$

$$= \Gamma^i_{ij}(x)\frac{\partial x^j}{\partial z^\alpha} + \frac{1}{\Delta}\frac{\partial \Delta}{\partial z^\alpha}$$

where Δ is the Jacobian, $|\partial x^j/\partial z^i|$ of the transformation. The last equations are the same as

$$\bar{\Gamma}^i_{i\alpha}(z) = \frac{\partial \log \gamma}{\partial x^j}\frac{\partial x^j}{\partial z^\alpha} + \frac{\partial \log \Delta}{\partial z^\alpha} = \frac{\partial}{\partial z^\alpha}\log \bar{\gamma}$$

where we are denoting by $\bar{\gamma}$ the function obtained by substituting the x's as functions of the z's in γ and multiplying by Δ. Hence the function γ is a scalar density and

$$\int \gamma\, dx^1\, dx^2 \ldots dx^n$$

may be taken as a definition of volume.

AFFINE GEOMETRIES OF PATHS POSSESSING AN INVARIANT INTEGRAL

BY L. P. EISENHART

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Read before the Academy, November 16, 1922

1. In the Riemann geometry volume is defined by the invariant integral $\int \sqrt{g}\, dx^1 \ldots dx^n$ where g is the determinant $|g_{ij}|$. If g' denotes the corresponding function when the coördinates are $x'^2, \ldots x'^n$, then

$$\sqrt{g'} = \sqrt{g}\,\Delta, \qquad (1.1)$$

where Δ is the Jacobian $\left|\frac{\partial x}{\partial x'}\right|$. When there exists for a geometry of paths a function g satisfying (1.1), we say that the geometry possesses an invariant integral, and g is called a *scalar density*. In a recent note (these PROCEEDINGS, **9**, p. 3) Professor Veblen showed that in an affine space for which $S_{ij} = 0$ a scalar density is defined by $\Gamma^\alpha_{\alpha i} = \partial \log \sqrt{g}/\partial x^i$; he calls

such a space *equiaffine*. It is the purpose of this note to show that *a necessary and sufficient condition that a goemetry of paths possess an invariant integral is that S_{ij} be the curl of a covariant vector*, and to derive some consequences of this theorem.

2. Let Γ^i_{jk} be the functions appearing in the equations of the paths (these PROCEEDINGS, Feb., 1922), then the functions Γ'^i_{jk} for a set of coördinates x' are given by

$$\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} + \Gamma^p_{qr} \frac{\partial x^q}{\partial x'^i} \frac{\partial x^r}{\partial x'^j} = \Gamma'^l_{ij} \frac{\partial x^p}{\partial x'^l}, \tag{2.1}$$

and the curvature tensor is defined by

$$B^p_{qrs} = \frac{\partial \Gamma^p_{qs}}{\partial x^r} - \frac{\partial \Gamma^p_{qr}}{\partial x^s} + \Gamma^p_{ar} \Gamma^a_{qs} - \Gamma^p_{as} \Gamma^a_{qr}. \tag{2.2}$$

By definition we have

$$S_{ij} = B^a_{aij} = \frac{\partial \Gamma^a_{aj}}{\partial x^i} - \frac{\partial \Gamma^a_{ai}}{\partial x^j}. \tag{2.3}$$

If equation (1.1) be differentiated, we have, in consequence of (2.1),

$$\begin{aligned} \frac{\partial \sqrt{g'}}{\partial x'^i} &= \frac{\partial \sqrt{g}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^i} \Delta + \sqrt{g}\, \Delta \frac{\partial^2 x}{\partial x'^i \partial x'^j} \frac{\partial x'^j}{\partial x^\alpha} \\ &= \Delta \sqrt{g} \left(\frac{\partial \log \sqrt{g}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^i} + \Gamma'^\alpha_{ai} - \Gamma^\alpha_{aq} \frac{\partial x^q}{\partial x'^i} \right), \end{aligned}$$

or, by means of (1.1),

$$\frac{\partial \log \sqrt{g'}}{\partial x'^i} - \Gamma'^\alpha_{ai} = \left(\frac{\partial \log \sqrt{g}}{\partial x^j} - \Gamma^\alpha_{aj} \right) \frac{\partial x^j}{\partial x'^i}. \tag{2.4}$$

From this equation it follows that

$$\partial \log \sqrt{g} / \partial x^j = \Gamma^\alpha_{aj} - \varphi_j, \tag{2.5}$$

where φ_j is a covariant vector. The conditions of integrability of equations (2.5) are

$$\frac{\partial \Gamma^\alpha_{aj}}{\partial x^i} - \frac{\partial \Gamma^\alpha_{ai}}{\partial x^j} = \frac{\partial \varphi_j}{\partial x^i} - \frac{\partial \varphi_i}{\partial \varphi^j}, \tag{2.6}$$

that is, S_{ij}, as defined by (2.3), is the curl of a vector, φ.

Conversely, if S_{ij} is the curl of a vector, φ_i, we have equations of the form (2.6) and (2.5) in each coördinate system, x and x', and consequently

equation (2.4) holds. If equations (2.1) be multiplied by $\frac{\partial x'^j}{\partial x^p}$ and summed for p and j, we obtain

$$\Gamma'^{\alpha}_{\alpha i} = \Gamma^{\alpha}_{\alpha q} \frac{\partial x^q}{\partial x'^i} + \frac{\partial \log \Delta}{\partial x'^i}.$$

By means of this relation, we obtain (1.1) from (2.4), and the theorem of § 1 is proved.

3. In a former paper (these PROCEEDINGS, Aug., 1922) the author considered spaces with corresponding paths and made the restriction in § 1 of that paper that s is the same for all paths. If this restriction be removed, the formulas (3.4) and (3.6) written in the form

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \varphi_k + \delta^i_k \varphi_j \quad (\delta^i_j = 1 \text{ for } i = j), \qquad (3.1)$$

where φ_i is a covariant vector, give the necessary and sufficient relations between the Γ's of two geometries of paths so that the paths are in one-to-one correspondence, and (3.7) gives the relations between s and $\bar{s}$ along corresponding paths. Equations (3.1) have been found by Weyl (*Gött. Nach.*, 1921), and also independently by Veblen (these PROCEEDINGS, Dec., 1922); they have interpreted them as the relations between the Γ's which yield the same paths in a space. Also they have remarked that each choice of the vector φ_i yields an affine space, whereas the paths define a projective space.

In my former paper it was shown that

$$\bar{S}_{ij} = S_{ij} + (n + 1)(\varphi_{ji} - \varphi_{ij}). \qquad (3.2)$$

From this equation it follows that if S_{ij} is the curl of a vector, and the vector $\varphi_i/(n + 1)$ is used in (3.1), then $\bar{S}_{ij} = 0$, that is, the space is equiaffine.

By definition the contracted curvature tensor R_{ij} is given by

$$R_{ij} = B^{\alpha}_{ij\alpha} \qquad (3.3)$$

From this it follows that

$$R_{ij} - R_{ij} = S_{ji} \qquad (3.5)$$

Hence the above result may be stated as follows:

Among the affine spaces possessing an invariant integral and having corresponding paths, one is equiaffine; for this space the contracted tensor is symmetric.

This result takes the place of the theorem stated in § 5 of my former paper, where an error was made in concluding that S_{ij} is the curl of a vector

for any geometry of paths (cf. my note on this point in the *Bull. Amer. Math. Soc.*, Dec., 1922).

4. The contracted tensor for a Riemann space is symmetric. Consequently if in (3.1) we replace Γ^i_{jk} by their expressions as Christoffel symbols of the second kind for a Riemann space, the functions $\overline{\Gamma}^i_{jk}$ define an affine space possessing an invariant integral. Hence:

The spaces with paths corresponding to the paths of a Riemann space possess an invariant integral.

CLOSED CONNECTED SETS WHICH ARE DISCONNECTED BY THE REMOVAL OF A FINITE NUMBER OF POINTS

BY JOHN ROBERT KLINE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA

Communicated, October 11, 1922

THEOREM A. *Suppose k is a positive integer and M is a closed connected point set in Euclidean space of two dimensions such that*

(1) *if $P_1, P_2, \ldots P_k$ are any k distinct points of M, then $M - (P_1 + P_2 + \ldots + P_k)$ is disconnected.*

(2) *if $Q_1, Q_2, \ldots Q_{k-1}$ are any $(k-1)$ distinct points of M, then $M - (Q_1 + Q_2 \ldots Q_{k-1})$ is connected.*

Under these conditions, M is a continuous curve.[1]

Proof.—Let us suppose that M is not connected im kleinen. Then there exists a point P belonging to M and a circle K with centre at P, such that within every circle whose centre is P there exists a point which does not lie together with P in any connected subset of M that lies entirely within K. Let $K_1, K_2 \ldots$ denote an infinite sequence of circles with centre at P and radius $r/2n$, where r is the radius of K. Let X_n denote a point within K_n such that X_n and P do not lie together in a connected subset of M which lies entirely within K. Let K' denote a circle with centre at P and radius $3r/4$. It follows with the use of a theorem due to Zoretti[2] that there is a closed connected set g_n, containing X_n and at least one point of K' not containing P and lying entirely within or on K'. It may easily be proved that there exist point sets $t_{n_1}, t_{n_2}, \ldots$ such that (1) for every i, t_{n_i} is a closed connected subset of M having at least one point on K' and at least one point on K_1 but no point within K_1 or without K', (2) for no values of i and j $(i \neq J)$ does t_{n_i} have a point in common with t_{n_j}. It follows that there exists an infinite sequence of integers $q_1, q_2, \ldots$ such that for every i, $q_{i+1} > q_i$ and a closed connected set t and a sequence of closed connected sets $k_{n_{q_1}}, k_{n_{q_2}} \ldots$ such that (1) for every i, $k_{n_{q_i}}$ is a subset of $t_{n_{q_i}}$, (2) each of

the sets t, k_{nq_1}, k_{nq_2}, . . . is a subset of $(K')' - K_1$ and contains at least one point on the boundary of K_1 and at least one point on the boundary of K', (3) if P_{nq_1}, P_{nq_2}, . . . is a sequence of points such that for every i, P_{nq_i} belongs to k_{nq_i}, then t contains every limit point of $P_{nq_1} + P_{nq_2}$, . . ., (4) if q'_1, q'_2, q'_3, . . . is an infinite sequence of distinct integers belonging to the set q_1, q_2 . . . and P is any point of t, then there exists an infinite sequence of points $P_{nq'_1}$, $P_{nq'_2}$, . . . such that for every i, $P_{nq'_i}$ belongs to $k_{nq'_i}$ and such that P is the sequential limit point of $P_{nq'_1}$, $P_{nq'_2}$, . . . Let P_1, P_2, . . . P_k be any k distinct points of t. Then $M - (P_1 + P_2 + \ldots + P_k) = M'_1 + M'_2$, two mutually separated sets.[3] If G is any point of t, then G is the sequential limit point of some G_{nq_1}, G_{nq_2}, . . . such that for every i, G_{nq_i} belongs to $k_{nq'_i}$. Of the two sets M'_1 and M'_2, one of them, which we shall denote by M_1, must contain an infinite subsequence of the point set G_{nq_1}, G_{nq_2}, . . . But M_2 denotes the other one of the sets M'_1 and M'_2. But as k_{nq_i} is a connected set, k_{nq_i} belongs to M_1. Hence it follows that $t - (P_1 + P_2 + \ldots + P_k)$ belongs to M_1.

It follows that $P_i + M_2$ is connected. For suppose $P_i + M_2 = H_1 + H_2$, two mutually separated sets and such that P_i is in H_1. Then $M - (P_{i+1} + P_{i+2} + \ldots + P_{i+k-1})$[4] $= M_2 + P_i + M_1 = H_1 + H_2 + M_1$. Clearly neither of the sets $(H_1 + M_1)$ and H_2 contains a limit point of the other one. Thus $H_i + M_2$ is connected.

Let us pick out any $k-1$ distinct points P_1, P_2 . . . P_{k-1} of t. Let G and H denote any two distinct points of $t - (P_1 + P_2 + \ldots + P_{k-1})$. Then $M - (G + P_1 + P_2 + \ldots + P_{k-1}) = S_1 + S_2$, two mutually separated set of which S_1 contains $t - (G + P_1 + P_2 + \ldots + P_{k-1})$. Likewise $M - (H + P_1 + P_2 + \ldots + P_{k-1}) = H_1 + H_2$, two mutually separated sets of which H_1 contains $t - (H + P_1 + P_2 \ldots P_{k-1})$. Now H_1 contains G. Hence the connected set $G + S_2$ is contained entirely in H_1 while H_2 is entirely in S_1. If Q is any point of $t - (P_1 + P_2 + \ldots + P_{k-1})$, then $M - (P_1 + P_2 + P_3 + \ldots + P_{k-1} + Q) = M_1 + M_2$, two mutually separated sets. Let t_Q denote that one of the sets M_1 and M_2 which does not contain $t - (P_1 + P_2 + \ldots + P_{k-1} + Q)$. Let $[t_Q]$ denote the set of sets thus obtained. If H_1 and H_2 are any two distinct points of $t - (P_1 + P_2 + \ldots + P_{k-1})$, t_{H_1} and t_{H_2} have no points in common. We can pick out of each of the sets $[t_Q]$ a definite point X_{t_Q}. We thus obtain a non-denumerable infinity of distinct points. One point B of $[X_{t_Q}]$ must be a limit of $[X_{t_Q}] - B$. Hence there exists a sequence B_1, B_2, B_3 . . . of $[X_{t_Q}] - B$ approaching B as its sequential limit point. Now B belongs to $t_{B'}$ for some point B' of $t - (P_1 + P_2 + \ldots + P_{k-1})$ while $B_i (i = 1, 2, 3, \ldots)$ belongs to $t_{B'_i}$ for some point B'_i of $t - (P_1 + P_2 + \ldots + P_{k-1})$. Now $M - (B' + P_1 + P_2 \ldots + P_{k-1}) = M_1 + t_{B'}$. Now M_1 contains $t - (B + P_1 + P_2 + \ldots + P_{k-1})$ and hence B_i is in M_1. Hence for every i, $B_i + t_{B_i}$ is in M_1. Hence

$B_1 + B_2 + B_3 \ldots$ is a subset of M_1. Thus B of t_B cannot be a limit point of $B_1 + B_2 \ldots$. Hence we are led to a contradiction if we suppose M is not connected im kleinen. Hence M is a continuous curve.

2. *The case where* $k = 1$.—We shall first prove several introductory lemmas.

Lemma A. Suppose M is a closed connected set such that (1) *M contains a point A such that* $M-A$ *is connected,* (2) *if B is any point of M different from A, then* $M-B$ *is the sum of two separated sets one of which is bounded. Then M is a ray*[5] *from A.*

Proof.—By methods similar to those of § 1, it follows that our set is a continuous curve. Let P denote any point of M, different from A. Then $M-P = M_1 + M_2$, two mutually separated sets of which M_1 is bounded. It follows by a theorem due to Mazurkiewicz[6] that M_2 is unbounded. Now $P + M_1$ is closed and connected. We shall show that the point A is in M_1. For suppose A were in M_2. Let $G_1 + G_2 + G_3 \ldots$ be a countable set such that every point of M either belongs to the set or is a limit point of the set. Now let G_{n1} be the point of lowest subscript of $G_1, G_2, \ldots$ in M_1. Let $M - G_{n1} = M_{11} + M_{12}$, two mutually exclusive separated sets of which M_{12} contains A. It follows that M_2 is a subset of M_{12}, that P is in M_{12} and M_{11} is a subset of M_1 and hence bounded. Let G_{n2} be the point of lowest subscript of $G_{n1+1}, G_{n1+2}, G_{n1+3} \ldots$ which is in $M_{1,1}$. Now $M - G_{n2} = M_{21} + M_{22}$, two mutually separated sets of which M_{22} contains A. It follows that M_{12} is a subset of M_{22}, G_{n1} is in M_{22} and M_{21} is a subset of $M_{1,1}$. Continue this process. For any i, $n_i > i$. The closed bounded sets $(G_{n1} + M_{1,1}), (G_{n2} + M_{2,1}), \ldots$ are such that for every i $(G_{ni+1} + M_{i+1,1})$ is a subset of $(G_{n_i} + M_{i,1})$. Hence they have at least one point X in common. As $X \neq A$, $M - X = M'_1 + M'_2$, two mutually separated sets of which M'_2 contains A. For every i, $G_{n_i} + M_{i,2}$ must be a subset of M'_2 while $M_{i,1}$ contains M'_1. Now let G_g be the point of lowest subscript in M'_1. Now G_g is in $M_{g-1,1}$ and is different from G_{ng}. Hence at this stage we did not choose the point of lowest subscript. Thus we have proved that A must be in M_1.

It may easily be proved that $M_1 + P$ is connected and connected im kleinen. Thus there is an arc AP from A to P lying wholly on $M_1 + P$. We shall now show that this arc AP contains the whole of $M_1 + P$. For suppose some point H of M_1 not on the arc. Then there is on $M_1 + P$ an arc HP from H to P. Let T denote the first point of the arc AP which is an arc of HP. Now as $H \neq A$, $M - H = G_1 + G_2$, two mutually separated sets of which G_2 contains P. Hence the connected set [$M_2 + P +$ arc AP] is on G_2, which is thus unbounded. But we have proved that A cannot belong to the unbounded set. Thus all points of $M_1 + P$ are on arc AP.

Let T be any bounded closed connected subset of M containing A. If S is any point of M not belonging to T, then $M - S = M_1 + M_2$, where M_1 contains A and is bounded. It follows that T belongs entirely to M_1. But $M_1 + S$ is an arc of M from A to S. If P is any point of arc $AS - T$, then there is an arc PS of M free entirely of points of T. As any bounded closed connected subset containing more than one point of an arc is still an arc the set T must be an arc from A to some point H of M_1. Now $M - H_1 = W_1 + W_2$, two mutually separated sets of which W_1 contains A. It follows that $W_1 + H_1$ is the set T while H_1 is the only point of T which is a limit point of $M - T$. Thus M is a ray from A.

Lemma B. In case $k = 1$, the set M of § 1 is unbounded.

Lemma C. In case $k = 1$, the set M of § 1 contains no simple closed curve.

For a proof of Lemmas B and C, compare Mazurkiewicz's article.

Theorem 1. In case $k = 1$ and P is any point of M, then there is an open curve 1 of M containing P.

Proof.—Suppose $M - P = M_1 + M_2$, two mutually separated sets. It follows that $P + M_i$ $(i = 1, 2)$ is a continuous curve. Hence by a theorem due to Kuratowski,[7] there is a ray from P which is a subset of $P + M_i$ $(i = 1, 2)$. The sum of these rays is an open curve through P.

Theorem 2. If P is any point of M then there are at most a finite number of rays $l_1, l_2 \ldots l_n$ of M such that l_i and l_j $(i = 1, 2 \ldots n, j = 1, 2 \ldots n$ and $i \neq j)$ have no point in common other than P.

Theorem 3. Within any circle there are but a finite number of points $P_1, P_2, \ldots P_n$ which are such that at each point P_i there are three or more rays of M having no point in common other than P_i.

Theorem 2 follows immediately by methods similar to those used previously in the paper. The proof of Theorem 3, omitted here, will be given in a subsequent paper.

3. The case $k = 2$.

Suppose k is an integer which is greater than or equal to 2 and M is a point set satisfying the conditions of Theorem A. Let $P_1 + P_2 \ldots P_k$ denote any k distinct points of M. Then $M - (P_1 + P_2 \ldots P_k) = M_1 + M_2$, two mutually separated sets. It is easy to show that $M_1 + (P_1 + P_2 + \ldots + P_k)$ is connected. We shall now show that $M_1 + (P_1 + P_2 + \ldots + P_k)$ is connected im kleinen. The proof is evident if R is any point of $M_1 + P_1 + P_2 \ldots P_k$ such that $R \neq P_i$ for $i = 1, 2, \ldots k$. Suppose $R = P_j$ and let K be any circle having P_j as centre. Put about R as centre a circle K' lying within K and such that there is within or on K' no point of the set $P_1 + P_2 + \ldots + P_{j-1} + P_{j-1} + \ldots + P_k$. As M is connected im kleinen there is a circle K'' with centre at R, lying within K' and such that if T is any point of M within K'', then T and R lie in a connected subset of M that lies within K'. It may easily be proved that T and R are the end-points of simple continuous arc of M lying entirely within K'.

Let H be any point of $M_1 + P_1 + P_2 + \ldots + P_k$ in K'', different from R. Then there is an arc $R\,X\,H$ from H to R belonging to M and lying entirely in K'. From the manner in which K' was chosen it follows that $R\,X\,H - R$ is subset of $M_1 + M_2$. Now $R\,X\,H - R$ is connected and one point of it, H, lies in M_1. Hence as M_1 and M_2 are mutually separated, M_2 can contain no point of $R\,X\,H - R$. Hence $R\,X\,H$ is a subset of $M_1 + P_1 + P_2 + \ldots + P_k$. Thus $M_1 + P_1 + P_2 + \ldots + P_k$ is a continuous curve. In like manner $M_2 + P_1 + P_2 + \ldots + P_k$ is a continuous curve.

There is in $M_i + P_1 + P_2 + \ldots + P_k$ $(i = 1, 2)$ a simple continuous arc $P_1 X_i P_2$. As these arcs have in common at most k points and have the same end-points, it is clear that there is a simple closed curve J which is a subset of $P_1 X_1 P_2 + P_1 X_2 P_2$, and thus a subset of M.

Let us suppose there are points of M not on J.

There are two cases:

Case I. There is a point T of M within J. Let A be any point of J. Then there is an arc $A\,X\,T$ of M. Let T' be the first point of $A\,X\,T$ going from T to A which is on J. Take $Q_1 + Q_2 + \ldots + Q_{k-1}$, $k - 1$ distinct points of J such that for every value of i from 1 to $k - 1$, $Q_i \neq T'$. Now $M - (T + Q_1 + Q_2 + \ldots + Q_{k-1}) = M_1 + M_2$, two mutually separated sets of which M_1 contains T'. As $TT' - T$ is connected all points of $TT' - T$ belong to M_1. Consider the set $T + Q_1 + Q_2 + \ldots + Q_{k-1} + M_2$. This set is a continuous curve and hence there is an arc TQ_1 lying entirely on this set. Let Q' be the first point of the arc TQ_1 going from T to Q_1 which is on J. Then arc $TT' +$ arc TQ' is an arc of $T'TQ'$ lying except for its end-points entirely within J. Now there must be points of M not on $J + T'TQ'$. For suppose that $M = J + T'TQ'$. There are two possibilities:

(*a*) $k = 2$. Let H and K be two points of J such that Q' and T' separate H and K on J. Then it is clear that $J - H - K + Q'\,TT'$ is connected, contrary to hypothesis.

(*b*) $k > 2$. Then $M - Q' - T' = Q'HT' + Q'KT' + Q'TT'$ and hence is not connected, which is contrary to assumption.

Hence there is a point E not on $J + Q'TT'$. There are two possibilities.

(a') E is without J. It follows there are $k - 1$ distinct points of M, $E_1, E_2, \ldots E_{k-1}$ without J. Consider $M - (E_1 + E_2 + \ldots E_{k-1} + T) = M'_1 + M'_2$, two mutually separated sets such that T' is in M'_1. It follows that all points of the connected set $J + Q'TT' - T$ are in M_1. Now $T + E_1 + \ldots + E_{k-1} + M_2$ is a continuous curve. Hence there is an arc TYE_1 lying entirely in $T + E_1 + E_2 + \ldots + E_{k-1} + M_2$. Hence TYE_1 has no point in common with J. But as T is within and E_1 is without J, the arc TYE_1 must have at least one point in common with J. Thus in case a' we are led to a contradiction.

(b') E is within J. Hence E is either (*i*) within $HQ'TT'H$ or (*ii*) within

$Q'KT'TQ'$. But in case (i) we have a point E within $HQ'TT'H$ and a point K without the same closed curve which is impossible by methods of case a' while in case (ii) we have E within $Q'KT'TQ'$ and H without the same closed curve.

Thus we are led to a contradiction if we suppose a point of M is within J. In exactly the same manner we can prove that no point of M is without J. Hence $M = J$. But as any simple closed curve is connected by the omission of any pair of its points, then k cannot be greater than 2.

Thus we are in a position to state the following theorems:

Theorem 4. If M is a closed connected point set in Euclidean space of two dimensions such that (1) if P is any point of M, then $M - P$ is connected, (2) if Q and R are any two distinct points of M, then $M - Q - R = M_1 + M_2$, two mutually separated sets. Under these conditions M is a simple closed curve.[8]

Theorem 5. If K is any positive integer greater than 2, then there is no closed connected set M satisfying the following conditions.

(a) If $P_1, P_2, + \ldots + P_k$ are any k distinct points of M then $M - (P_1 + P_2 + \ldots + P_k)$ is disconnected.

(b) If $Q_1, Q_2, + \ldots + Q_{k-1}$ are any $k - 1$ distinct points of M, then $M - (Q_1 + Q_2 \ldots Q_{k-1})$ is connected.

[1] The term *continuous curve* is here used in the sense suggested by Professor R. L. Moore, who applies this term to sets which are closed, connected and connected im kleinen. Cf. R. L. Moore, *Trans. Amer. Math. Soc.*, **21** (1920), p. 347. A set of points is said to be *connected im kleinen* if for every point P of M and every circle K with centre at P there exists a circle $K_{k,P}$ within K and with centre at P such that if X is a point of M within $K_{k,P}$ then X and P lie together in some connected subset of M that lies entirely within K. Cf. Hans Halm, *Ueber die Allgemeinste ebene Punktmenge, die stetiges Bild einer Strecks ist, Jahrb. der Deut. Math. Ver.*, **23** (1914), pp. 318–22.

[2] L. Zoretti, *Sur les fonctions analytiques uniformes, J. Math. Pures App.*, (ser. 6) **1**, (1905), p. 10.

[3] Two points sets are said to be *mutually separated* if neither contains a point or limit point of the other one. Cf. R. L. Moore, loc. cit., p. 341.

[4] It is understood that subscripts are reduced modulo k.

[5] An *open curve* is a closed and connected point set which is separated into two connected sets by the omission of any one of its points. If P is a point of an open curve M, the point set obtained by adding P to either of the two sets into which M is separated by the omission of P is called a *ray*. Prof. R. L. Moore proves that a ray is a continuous curve M containing a point A such that every bounded continuous subset of M that contains A has just one boundary point with respect to M. Cf. R. L. Moore, loc. cit., p. 347.

[6] S. Mazurkiewicz, *Un theoreme sur les lignes de Jordan, Fundamenta Math.*, **2**, p. 119.

[7] C. Kuratowski, *Quelques properties topologiques de la semi-droite, Ibid.*, **3**, p. 61.

[8] Compare H. Tietze, *Ueber Stetige Kurven und Jordansche Kurven, Math., Zts.*, **5** (1919), p. 289. Tietze makes the assumption that the set is connected im kleinen. This is really a consequence of his other conditions.

SOME EXTENSIONS IN THE MATHEMATICS OF HYDROMECHANICS

BY ROBERT S. WOODWARD

CARNEGIE INSTITUTION OF WASHINGTON, D. C.

Read before the Academy, April 25, 1922

Present Status and Needs of Hydrokinetics.—Precise description of fluid motion when viscosity is taken into account is a matter of much complexity. No less than twenty symbols and fifteen equations are required for a complete specification of the linear and angular movements of a fluid element, of its expansion and contraction, and of the changes in pressure and density to which it is subject. The formation of what Poinsot called an "image sensible" of these characteristics is a matter of corresponding difficulty which has taxed the resources of the ablest analysts from the time of Euler down to the present day. The nature of the difficulty seems to lie in an inadequacy of our mathematical machinery rather than in defects of our physical concepts. The following abstract, therefore, aims to suggest some extensions and improvements in that machinery, with the hope that greater generality and uniformity may be ultimately realized in the treatment of concrete problems in this branch of physics.

The general equations of fluid motion are commonly written in the following simplest forms, namely:[1]

$$\begin{aligned}
\rho \frac{du}{dt} + \frac{\partial p}{\partial x} &= \rho X + \frac{1}{3}\mu \frac{\partial \theta}{\partial x} + \mu \Delta^2 u, \\
\rho \frac{dv}{dt} + \frac{\partial p}{\partial y} &= \rho Y + \frac{1}{3}\mu \frac{\partial \theta}{\partial y} + \mu \Delta^2 v, \\
\rho \frac{dw}{dt} + \frac{\partial p}{\partial z} &= \rho Z + \frac{1}{3}\mu \frac{\partial \theta}{\partial z} + \mu \Delta^2 w.
\end{aligned} \tag{1}$$

In these x, y, z are the rectangular coördinates of any element of the fluid at the time t; u, v, w are the linear velocity components of the element; p is the internal stress and ρ is the density of the fluid at x, y, z, at the time t; X, Y, Z are the force components per unit mass of the fluid at x, y, z; μ is the coefficient of viscosity of the fluid; and θ is the time rate of expansion of the element at x, y, z at the time t. This rate is given by the equation expressing the conservation of the mass of the element, that is, by

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{d \log \rho}{dt} \tag{2}$$

The symbol Δ^2 which appears in the second members of (1) is defined, for example, by

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

and the operation here indicated is now commonly called the Laplacian of u.

Along with the equations (1) and (2) three others are needed to express the relations between the linear velocity components u, v, w and the angular velocity components (or spin components) of the fluid element. These latter components with reference to the axes of X, Y, Z, respectively, are given by

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad 2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{3}$$

and these are subject to the obvious condition

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0. \tag{4}$$

Moreover, three additional equations, introducing four new functions (potentials), are required to express u, v, w in forms to meet the requirements of the conditions (2) and (4) These equations are

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} + \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} \\ v &= \frac{\partial \varphi}{\partial y} + \frac{\partial V}{\partial z} - \frac{\partial W}{\partial x} \\ w &= \frac{\partial \varphi}{\partial z} + \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} \end{aligned} \tag{5}$$

in which φ is the velocity potential, representing the irrotational part of the motion, and U, V, W may be called the spin potentials, since they represent the rotational part of the motion, these two parts being independent of one another.

Finally the conditions (2) and (4) lead to the further relations[2]

$$\begin{gathered} \Delta^2 \varphi = \theta, \\ \Delta^2 V = -2\xi, \qquad \Delta^2 V = -2\eta, \qquad \Delta^2 W = -2\zeta, \\ \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0. \end{gathered} \tag{6}$$

Much embarrassment hinges on the application of these equations (6), which seem to have figured thus far chiefly in the rôle of "existence the-

orems;" and it is one of the objects of the following paragraphs to indicate how these relations may be utilized in the solution of concrete problems.

Extension of Relations between u, v, w and ξ, η, ζ.—Equations (3) suggest the following obvious relations derived from (3) and (2):

$$\begin{aligned} 2\left(\frac{\partial\zeta}{\partial x} - \frac{\partial\eta}{\partial z}\right) &= \frac{\partial\theta}{\partial x} - \Delta^2 u, \\ 2\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right) &= \frac{\partial\theta}{\partial y} - \Delta^2 v, \\ 2\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right) &= \frac{\partial\theta}{\partial z} - \Delta^2 W. \end{aligned} \tag{7}$$

Operating on (7) in a similar manner there result

$$\begin{aligned} \frac{\partial\Delta^2 w}{\partial y} - \frac{\partial\Delta^2 v}{\partial z} &= 2\Delta^2\xi, \\ \frac{\partial\Delta^2 u}{\partial z} - \frac{\partial\Delta^2 w}{\partial x} &= 2\Delta^2\eta, \\ \frac{\partial\Delta^2 v}{\partial x} - \frac{\partial\Delta^2 u}{\partial y} &= 2\Delta^2\zeta. \end{aligned} \tag{8}$$

Again, there follow from (8) and (2)

$$\begin{aligned} 2\left(\frac{\partial\Delta^2\zeta}{\partial y} - \frac{\partial\Delta^2\eta}{\partial z}\right) &= \frac{\partial\Delta^2\theta}{\partial x} - \Delta^2\Delta^2 u, \\ 2\left(\frac{\partial\Delta^2\xi}{\partial z} - \frac{\partial\Delta^2\zeta}{\partial x}\right) &= \frac{\partial\Delta^2\theta}{\partial y} - \Delta^2\Delta^2 v, \\ 2\left(\frac{\partial\Delta^2\eta}{\partial x} - \frac{\partial\Delta^2\xi}{\partial y}\right) &= \frac{\partial\Delta^2\theta}{\partial z} - \Delta^2\Delta^2 w. \end{aligned} \tag{9}$$

The equation (3) shows that when $\xi = \eta = \zeta = 0$, or when there is no rotation, the well-known velocity potential exists, or that

$$u = \partial\varphi/\partial x, \quad v = \partial\varphi/\partial y, \quad w = \partial\varphi/\partial z, \quad \Delta^2\varphi = \theta. \tag{10}$$

Similarly, equations (7) show that when θ is constant and when u, v, w are each harmonic,[3] a potential F exists, such that

$$2\xi = \partial F/\partial x, \quad 2\eta = \partial F/\partial y, \quad 2\zeta = \partial F/\partial z, \quad \Delta^2 F = 0. \tag{11}$$

Likewise, equations (8) show that when ξ, ζ, η are harmonic, or when $\Delta^2\xi = \Delta^2\eta = \Delta^2\zeta = 0$ a generating potential, G, obtains so that

$$\Delta^2 u = \partial G/\partial x, \qquad \Delta^2 v = \partial G/\partial y, \qquad \Delta^2 w = \partial G/\partial z, \qquad \Delta^2 G = \Delta^2\theta. \tag{12}$$

Further, when $\Delta^2\theta$ is constant and when $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$ are harmonic (or when u, v, w are preharmonic as defined below) equations (9) show that $\Delta^2\xi$, $\Delta^2\eta$, $\Delta^2\zeta$ are harmonic and derivable from a function H, such that

$$\Delta^2\xi = \partial H/\partial x, \qquad \Delta^2\eta = \partial H/\partial y, \qquad \Delta^2\zeta = \partial H/\partial z, \qquad \Delta^2 H = 0. \tag{13}$$

Harmonic and Preharmonic Functions.—The functions which appear under the symbol Δ^2 in equations (6) to (13) are of surpassing importance and interest in mathematical physics. Functions Q, say, satisfying the equation of Laplace, namely, $\Delta^2 Q = 0$, have been very extensively studied, especially during the past half century. They are now generally called harmonic functions, or, briefly, harmonics, following the exposition of their properties and uses given by Thomson and Tait in their *Natural Philosophy*,[4] which is probably still the richest mine in this domain, though somewhat difficult of access. What is given below is only an extension of the suggestions found in their great work.

A harmonic function, Q, is any homogeneous function of x, y, z satisfying the equation $\Delta^2 Q = 0$. From the indications of Thomson and Tait in the work cited it follows that the harmonics of negative integral degree are given, omitting possible constant factors, by the formulas

$$Q_{-(i+1)} = \frac{\partial^i \frac{1}{r}}{\partial x^j \partial y^k \partial z^i} \tag{14}$$

$$r^2 = x^2 + y^2 + z^2$$

$$i = j + k + z = 0, 1, 2, \ldots$$

Thus, for

$$i = 0, \qquad Q_{-1} = \frac{1}{r};$$

$$i = 1, \qquad Q_{-2} = -\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3};$$

$$i = 2, \qquad Q_{-3} = -\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right), \text{ etc.}$$

Similarly, the harmonics of positive integral degree are given by the formulas

$$Q_i = r^{2i+1} \frac{\partial^i \frac{1}{r}}{\partial x^j \partial y^k \partial z^i}, \qquad (15)$$

$$r^2 = x^2 + y^2 + z^2$$

$$i = j + k + z = 0, 1, 2, \ldots$$

Thus, for

$$i = 0, \quad Q_0 = +1;$$

$$i = 1, \quad Q_1 = -x, -y, -z;$$

$$i = 2, \quad Q_2 = +3xy, 3yz, 3zx,$$

$$-(r^2 - 3x^2), -(r^2 - 3y^2) - (r^2 - 3z^2).$$

Corresponding to every harmonic there exists another function of nearly equal importance, namely, a function whose Laplacian is the given harmonic. If the latter is denoted by Q, as above, and if P is the corresponding function,

$$\Delta^2 P = Q, \text{ and } \Delta^2 \Delta^2 P = \Delta^2 Q = 0. \qquad (16)$$

I have ventured to call the function P, defined by (16), a preharmonic, although the term biharmonic has been used by European mathematicians to designate such functions, without special reference, however, so far as I am aware, to their correspondence with harmonics.

The preharmonics corresponding to the harmonics given by (14) are readily found. For, since

$$\frac{1}{r} = \frac{1}{2} \Delta^2 r,$$

(16) gives

$$\Delta^2 P_{-(i-1)} = \frac{1}{2} \frac{\partial^i \Delta^2 r}{\partial x^j \partial y^k \partial z^i},$$

whence

$$P_{-(i-1)} = \frac{1}{2} \frac{\partial^i r}{\partial x^j \partial y^k z^i} + H, \qquad (17)$$

$$i = j + k + z = 0, 1, 2, \ldots,$$

wherein H is any harmonic or the sum of any number of such. In each case the degree of the preharmonic is higher by 2 than the degree of the harmonic. Thus, to illustrate, for

$$i = 0, \quad Q_{-1} = +\frac{1}{r}, \text{ and } P_{+1} = \frac{1}{2} r + H;$$

$$i = 1, \quad Q_{-2} = -\frac{x}{r^3}, \text{ and } P_0 = \frac{1}{2}\frac{x}{r} + H;$$

$$i = 2, \quad Q_{-3} = -\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right), \text{ and } P_{-1} = \frac{1}{2}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + H.$$

Similarly, the preharmonics corresponding to the harmonics of positive integral degree, or to those defined by (15), are given by the solution of

$$\Delta^2 P_{i+2} = Q_i,$$

and this is found to be

$$P_{i+2} = \frac{r^2 Q_i}{2(2i+3)} + H, \tag{18}$$

in which H is any harmonic or the sum of such. To illustrate, by a single example drawn from each of the groups for which $i = 0, 1, 2$, respectively, and omitting constant factors which would enter in any applications, when

$$i = 0, \quad Q_0 = +1, \text{ and } P_2 = +\frac{1}{6} r^2 + H;$$

$$i = 1, \quad Q_1 = -x, \text{ and } P_3 = -\frac{1}{10} r^2 x + H;$$

$$i = 2, \quad Q_2 = +3xy, \text{ and } P_4 = +\frac{3r^2 xy}{14} + H.$$

It is seen that by aid of equations (17) and (18) the troublesome triple integrals by which φ, U, V, W of (6) are commonly expressed may be avoided whenever θ, ξ, η, ζ are known harmonics; and when these latter quantities are known preharmonics, the relations $\Delta^2\Delta^2\varphi = \Delta^2\theta$, etc., become available. In general it will be found that the relations between the velocity and the spin components, and their Laplacians, given by equations (7) to (13), along with the harmonics and preharmonics given by (14) to (18), serve to remove many of the difficulties of this complex branch of mathematical physics.

[1] See, for example, Lamb's *Hydrodynamics*, edition of 1895, p. 515.

[2] First given in the classic memoir of Helmholtz of 1858, *Crelle's Journal*, Vol. LV.

[3] As explained below, a harmonic function, Q say, is one whose Laplacian is zero, or briefly $\Delta^2 Q = 0$.

[4] *Treatise on Natural Philosophy*, by Thomson and Tait, Cambridge, 1879–1883, Vol. I, Part I, Appendix B, pp. 171–218.

HYDROHEPATOSIS, A CONDITION ANALOGOUS TO HYDRONEPHROSIS

BY PHILIP D. McMASTER AND PEYTON ROUS

ROCKEFELLER INSTITUTE OF MEDICAL RESEARCH, NEW YORK

Read before the Academy, November 15, 1922

The changes which follow obstruction to the common bile duct in human beings are highly various; and clinical and autopsy records yield few clues to the factors responsible for this diversity. We have attempted to come at them by animal experiment.

The fluid found in the distended ducts above an obstruction that has endured some time may be heavily loaded with bile pigment and mucus, green-black and tarry, or so wholly devoid of these constituents, despite a pronounced jaundice of the tissues in general, that it is colorless, limpid and watery. All gradations between such extremes are met with. We have shown in previous papers[1,2] that in the absence of infection all are traceable to the differing influences of the gall-bladder and ducts upon the bile first pent in the channels. The ducts elaborate a secretion of their own, which, thin and colorless, tends gradually to replace the bile; whereas the gall-bladder, by virtue of special abilities to concentrate the bile[2] and add mucus thereto, acts to fill the passages with one so thick and inspissated that the secretion just mentioned dilutes it but slowly and never replaces it perfectly. When the gall-bladder fails to exert its characteristic influences, as happens sometimes after injury to the organ, the fluid collecting in it, and in the ducts as well, is a colorless "white bile" similar to that contained in obstructed ducts unconnected with the *receptaculum felleae*.

These are the findings after total obstruction. The changes taking place when the liver continues to secrete into the proper channels but against a pressure obstacle, have not only theoretical interest but a practical one as bearing upon the development of gallstones. We have studied them with the aid of a method whereby in dogs the common duct, and very probably the duct of any gland lying within the peritoneal cavity, may be permanently intubated and the secretion collected. The method depends upon the ability of the omentum to limit the spread of infection. When a rubber tube is connected with an intraperitoneal duct and led directly through the abdominal wall, it ordinarily comes away within a few days as the result of an ascending purulence. But if the tube be so bent upon itself that a considerable stretch of it, sheathed in omentum, is interposed between the duct and the exit wound, infection does not travel along it and it remains permanently in place. The pressure obstacle necessary to the observations was found, in some instances, in the tissue adhesions that developed about a balloon for bile collection which

was interpolated in that part of the tube system lying within the abdomen. As the adhesions gradually thickened and contracted, they brought to bear an increasing resistance to the expansion of the balloon. In other instances, in which the balloon was placed outside of the animal in a basket moored to the skin, small gallstones forming in the lumen of the glass canula placed in the common duct interfered as they enlarged with the passage of the bile. Whichever the type of obstacle the same series of changes in the bile took place. From day to day as the pressure against which the fluid was secreted became greater, it gradually lost the characters distinguishing it as bile. While still fairly copious, it contained less and less bilirubin, cholesterin and bile salts. Only faint traces of these substances were present in the last scanty specimens obtained before obstruction became complete. The final fluid distending the ducts was a "white bile" such as has already been mentioned, that is to say, no bile at all, in the ordinary sense of the term.

A comparison of the liver changes with those occurring in other glands after duct obstruction has proved enlightening. These changes have been best studied in the kidney. In this organ the pressure under which the retained secretion is held interferes with the venous flow—and it does this in the liver as well, a point which we have brought out in a previous paper.[3] Both organs continue to secrete; but the secretion is now turned back into the body. There is in both a progressive connective tissue proliferation, and an accompanying parenchymal atrophy. The atrophy may become complete when only one kidney is affected; but when the outflow of urine from all of the renal tissue is interfered with death ensues long before the alterations progress so far. Similarly, obstruction of the bile ducts from but a portion of the liver leads to a complete parenchymal atrophy in the region concerned;[4] whereas if the common duct, which drains all of the organ, be occluded, the individual dies at a time when the hepatic changes are still relatively slight.

A striking early change in the kidney is sacculation, and the sac may come to contain several litres of a watery fluid devoid of the characters of urine—hence the name for the condition, *hydronephrosis*. Obstruction of the duct from the liver leads to a marked distention of the bile channels, but the organ itself does not bag out, a difference scarcely surprising when one considers that the maximum pressure developed upon biliary obstruction is extremely low (350 mm. of the bile itself, in the dog), whereas on renal obstruction it is high (from 650 to 900 mm., in terms of bile). The fact that the fluid ultimately found in the distended ducts from the liver has, in the absence of gall-bladder influence, none of the characters of bile, has been sufficiently indicated. When secretion takes place against a pressure obstacle there is a reduction both in the total output and in the percentage output of some substances, exactly as with the kidney.[5]

Whether, as in the case of the latter organ, the percentage output of certain other substances is increased, remains to be determined.

From this comparison it will be seen that the changes consequent upon biliary obstruction and upon renal obstruction, respectively, are essentially similar in type. They are indeed the changes which follow obstruction to the ducts of glands in general; and the differences between the renal and hepatic manifestations of them are traceable merely to incidental peculiarities of the glands concerned. These facts would long ago have been recognized, were it not for the complicating activities during biliary obstruction of an extraneous organ, the gall-bladder. As indicating the essential likeness of the hepatic to the renal changes, we would suggest that the term *hydrohepatosis* be applied to the condition found after obstruction to the outflow of bile. When the contents of the obstructed ducts have been altered through gall-bladder activity, the hydrohepatosis is *concealed*, in a very real sense; but when this has not happened, and the distended ducts are in consequence filled with "white bile," there is a *manifest hydrohepatosis*. But the terms have little importance as compared with the understanding that they are supposed to convey.

[1] Rous, P., and McMaster, P. D., *J. Exper. Med., Baltimore*, 1921, **34**, 47.
[2] Rous, P., and McMaster, P. D., *Ibid.*, 1921, **34**, 75.
[3] Rous, P., and Larimore, L. D., *Ibid.*, 1920, **32**, 249.
[4] Nasse, *Verhandl. deutsch. Ges. Chir.*, 1894, Berlin, xxiii, Pt. 2, 525.
[5] Cushny, A. R., *The Secretion of the Urine*, London, New York, Bombay, Calcutta, and Madras, 1917.

NOVOCAINE AND CURARISATION

BY H. LAUGIER AND R. LEGENDRE

FACULTÉ DES SCIENCES, AND ÉCOLE DES HAUTES ÉTUDES, PARIS

Communicated, November 20, 1922

In a paper published in these PROCEEDINGS, in April, 1921, Fulton proposed novocaine as a "convenient substitute for curare."

He says that he observed that, when the sciatic-gastrocnemius preparation of a frog is immersed in a 2.5% solution of novocaine, this solution is without effect on the nerve but suppresses the excitability of the muscle. A dye made by linking novocaine with a benzene nucleus has the same physiological action, and stains the muscular nucleus but not the nerve fibers or the end-plates. The author concludes that novocaine acts specifically on some portion of the neuro-muscular mechanism located beyond the end-plates.

Such observations, statements and conclusions are direct contradictions of the many physiological and morphological facts that we have previously observed and published.

Physiologically,[1] novocaine produces considerable alteration of nerve fiber's ability to conduct even in such weak solutions as 0.1 in 1000 which is 250 times more diluted than that which Fulton declares inactive. A solution in physiological salt water of novocaine (0.1 in 1000) produces a rapid variation of the two characteristics of excitability : increasing the rheobase and lessening the chronaxie. It is known that Lapicque[2] has given the name of *rheobase* to the lowest intensity of a suddenly established current that will provoke the first visible muscular contraction, and the name *chronaxie* to the time that is required for a current of double intensity to produce the same response.

If there might be some doubt regarding the meaning of the rheobasic increase, there can be no ambiguity in the interpretation of chronaxie. The variations of chronaxie, as numerous experiments on localisation of excitation show, produce a change at the point excited, that is to say, in the part of the nerve near the cathode. Therefore, novocaine acts upon the nerves.

These physiological observations entirely agree with microscopic observations of the living nerve. Under the influence of novocaine, important morphological modifications appear in the intact fiber: changes in the appearance of the myeline, which becomes more bright, then swells, and produces, on the inner face of the sheath, protuberances which finally occupy a large section of the cylindraxis.

Such modifications are reversible and proceed in similar ratio to the variations in excitability.[3]

From these concording physiological and histological observations, it is shown that novocaine exerts a pronounced action on the nerve fiber.

There is nothing surprising in the phenomenon of curarisation mentioned by the author, if one examines it according to the latest theories. In fact, Lapicque has shown[4] that, under normal conditions, the nerves and muscles are in physiological accord and have the same chronaxie (homochronism). When, under any influence, the chronaxie of both muscle and nerve becomes markedly different (heterochronism), the excitation no longer passes from the nerve to the muscle (curarisation). Thus, whenever a poison acts selectively on the nerve or muscle or even acts with a different speed on one and the other, we can observe during the time of systemic intoxication a state of curarisation. This is the action that novocaine produces, as a considerable number of poisons called *curarisants*, or even simply a very concentrated solution of NaCl.

[1] R. Deriaud & H. Laugier, *C. R. Soc. Biol., Paris*, **85**, 1921 (324–328) (with remarks of R. Legendre).

[2] L. Lapicque, *C. R. Soc. Biol., Paris*, **67**, 1909.

[3] L. Lapicque & R. Legendre, *C. R. Acad. Sci., Paris*, **158**, 1914 (1592–1595); *Bull. Mus. nat. Hist. Natur., Paris*, **20**, 1914 (367–271); *J. Physiol. Pathol. gén., Paris*, **20**, 1922 (163–172).

[4] L. & M. Lapicque, *C. R. Soc. Biol., Paris*, **65**, 1908, and **68**, 1910 (1007–1010).

UNDERNUTRITION AND ITS INFLUENCE ON THE METABOLIC PLANE OF STEERS

BY FRANCIS G. BENEDICT AND ERNEST G. RITZMAN

NUTRITION LABORATORY OF THE CARNEGIE INSTITUTION OF WASHINGTON, BOSTON, MASS., AND NEW HAMPSHIRE AGRICULTURAL EXPERIMENT STATION, DURHAM, NEW HAMPSHIRE

Read before the Academy, November 14, 1922

A study of the influence of prolonged undernutrition upon a group of men, in which it was found that the recovery of these men with subsequent liberal amounts of food was very rapid,[1] and a close study of the feeding habits of wild animals made it appear that an investigation with regard to the influence of reduced rations upon large ruminants (steers) would be not only of scientific but of economic importance. At the Agricultural Experiment Station, Durham, New Hampshire, therefore, eleven adult steers were subjected for a period of about $4^1/_2$ months to a reduction in ration approximating one-half of their original maintenance requirement. A control group of three steers were accorded exactly the same barn treatment but received sufficient hay to maintain them at nearly constant body weight. At the close of the underfeeding period eleven of the steers were given fattening rations and three were put upon grass.

Because of the difficulties of estimating from live weights the true changes in the mass of body tissue, a special respiration chamber was designed and constructed to study the carbon-dioxide production of these steers at different stages of the observation. Numerous changes in body weight occur daily with animals of this size, explainable to a great extent by the transitional changes in the large amount of intestinal fill or ballast, which with an adult steer weighing 500 or 600 kilograms may amount to more than 20 per cent of its body weight. With the curtailment of ration there was at first a rapid loss in weight, due to changes in fill, and subsequently a slow but steady loss, due to drafts upon body material. During the last few weeks of undernutrition the weights of practically all the steers remained at essentially a constant level. On refeeding, when double the quantity of hay was given for a few weeks, a correspondingly rapid increase in weight took place, but from the caloric value of the hay ration, its digesti-

bility, and amount, this increase could by no means be ascribed to deposited tissue but was undoubtedly due to changes in fill. Girths at the chest paunch, and hind flank support this deduction, since the chest circumference is only negligibly affected by sudden changes in fill.

The amounts of water consumed varied widely from day to day, i.e., from 0 to 80 kilograms. But with all the steers the amount of water consumed per kilogram of water-free substance in feed was found to be nearer 2.5 kilograms than the commonly accepted value of 4 or 5 kilograms.

The amount of urine excreted was but little affected by the increase in ration and the resulting increase in water consumption. It was found that while highly nitrogenous rations did stimulate to a larger secretion of urine, relatively large increases in the nitrogen intake could be made without producing this effect. The amount of water carried off with undigested feed residues was in general proportional to the amount of water-free substance in the feces. During the periods of lowered rations the feces of all the steers were distinctly harder and in smaller masses, and yet the analyses showed that their moisture content was but 2 or 3 per cent less. The energy of the feces was remarkably constant under all feeding conditions, i.e., on the average 4.778 calories per gram of water-free substance or 5.210 calories per gram of water- and ash-free substance.

The digestibility of a single feedstuff like hay was practically unaltered either by the amount of hay fed within very wide limits or by the nutritive plane of the animal.

During the first few days of undernutrition the steers showed a restlessness and craving for greater bulk in the feed, but as soon as the intestinal tract became adjusted to the decreased fill, the animals became very quiet, and although they moved around more slowly and with less vigor than the control steers, in an exercise yard they showed astonishing activity for animals on these supposedly weakening rations, even after $4^1/_2$ months of submaintenance feeding.

The pulse rates decreased from an average of not far from 44 to 28 beats per minute or below, one steer having a pulse rate on one day as low as 20 beats per minute.

Metabolism measurements made under standard (but not basal) conditions, that is, with the steer awake and standing with minimum muscular activity, 24 hours after the last ingestion of feed, showed that in general the maintenance level of metabolism was 2150 calories per 24 hours per square meter of body surface. The undernourished animals showed a low metabolic level of 1475 calories. With realimentation the metabolism increased to not far from 2200 calories.

With certain of the steers complete nitrogen balances were obtained. From these data and from data regarding the metabolizable energy in the feed, the standard metabolism, and the computed 24-hour metabolism, it

is estimated that these adult steers, when subjected to a 50 per cent reduction in rations consisting of hay only, went through the winter of 140 days with a loss of 1300 grams of nitrogen and approximately 52 kilograms of fat. That this loss was not inimical to the subsequent development of the steers is shown by the fact that on pasture and with concentrates and hay these animals were readily fattened for market.

The investigation here reported is being supplemented by an extensive series of observations during complete fasting. The results of the work on undernutrition are about to be published by the Carnegie Institution of Washington.

[1] The results of this study were presented before the Academy on April 22, 1918. See Benedict, Miles, Roth, and Smith, *Proc. Nat. Acad. Sci.*, 1918, **4**, p. 149.

A QUANTUM THEORY OF OPTICAL DISPERSION

By C. G. Darwin, F.R.S.

Normal Bridge Laboratory of Physics, California Institute of Technology

Communicated, December 1, 1922

When a theory is framed trying to explain a discrepant system of facts, it is a necessary process of thought to take some branch of the theory as more completely true than the rest, and to adjust the remaining parts in such a way that they will fit in with this base, though they may still conflict with one another. This has certainly been true of the quantum theory; the speculations connected with it have as their base the law of the conservation of energy. Now a critical examination of fundamentals does not by any means justify this faith. It is of course a fact of observation that in the gross energy is conserved, but this only means an averaged energy; and as pure dynamics has failed to explain many atomic phenomena there seems no reason to maintain the exact conservation of energy, which is only one of the consequences of the dynamical equations. Indeed it is hardly too much to say that had the photoelectric effect been discovered a century ago, it is probable that no one would ever have suggested that the status of the first law of thermodynamics was in any way different from that of the second. On the other hand, Bohr's theory, and especially Sommerfeld's extension of it, have given great encouragement to the belief that in dynamics lay the way to the complete truth, so that in consequence of the triumphs of that theory there has been little thought in other directions. Another impediment is that our whole ideas are saturated with the principles of energy, so that denying it leaves hardly any foundation from which to start.

Now there is another field of phenomena which forms a consistent whole, but which at present only fits into the quantum theory with a good deal of difficulty, and that is the wave theory of light. Interference and diffraction are completely explained by a wave theory, and it would seem almost impossible to devise any really different alternative which would account for them. Here is a base which seems to be free from the objections which attach to energy, and I have therefore been examining the consequences of fitting it in with those parts of the Bohr theory which seem to be most completely established. The result is what I believe to be a satisfactory theory of dispersion—one of the weakest points in the quantum theory*—and a great promise of future extensions in other directions.

We shall assume, then, that the wave theory gives a correct account of events *outside matter*, and it is convenient to take over the terminology of the electromagnetic theory, provided we remember that "electric force" is only to mean "light vector" and that we are not prescribing how the electric force will affect the behavior of atoms or electrons. The assumption brings with it, of course, the exact conservation of energy in the aether; it is in interchanges with matter that it need not be conserved. When a wave passes over matter there is a mutual influence and without any inquiry into what happens to the matter, we can say that it is inconceivable that the effect on the aether should be anything but in the form of an expanding spherical wave. Every such wave can be described in terms of spherical harmonics, and the simplest is the one corresponding to the harmonic of zero order. In this the electric force vanishes at two poles and is elsewhere along the lines of longitude proportional to the cosine of the latitude, while the magnetic force lies in the circles of latitude. This is the type of wave given in the classical theory by a Hertzian doublet vibrating in a line and it proves unnecessary for our theory to postulate that any more complicated type is emitted by the atom. If x is the direction of the pole of the wave, then at $x, y\ z$ at a great distance r from the atom the wave is given by:

$$E_x = -\frac{r^2 - x^2}{r^3} f(t - r/c),\ E_y = \frac{xy}{r^3} f(t - r/c),\ E_z = \frac{xz}{r^3} f(t - r/c). \quad (1)$$

Next, borrowing from the Bohr theory we shall assume that when an atom is struck by a wave, there is a certain chance that the atom should emit a secondary wave of the above type. With these assumptions it is possible to argue inductively from the observed fact that if incident waves are superposed the result can be found by an addition of their effects and from the known form of the dispersion formula. There is no need to give the argument but only its result. The complete statement of this for unpolarised

waves is rather more complicated, but the essential points of the theory are fully represented in what follows.

When a wave, polarised so that the electric force is along x, strikes an atom at the origin there is a chance $A_n(\partial E_x/\partial t)dt$ that in the time dt it will excite the atom to emit a spherical wave of the type (1) with f of the form $\alpha_n e^{-\lambda_n t}\cos k_n t$. *Here* A_n, α_n λ_n *and* k_n *depend only on the nature of the atom and not at all on the incident force.* λ_n is supposed to be small. Of course $\partial E_x/\partial t$ may be negative; in this case we shall suppose that there is a chance $A_n(-\partial E_x/\partial t)dt$ for the emission of a wave $-f$. We shall be able to treat both cases together and need not make the distinction. The subscript n indicates that we suppose there are several different ways in which the atom may be excited, each with a separate chance for it.

Consider a simple case, a monochromatic wave polarised along x and advancing along z, which strikes a group of N atoms at the origin. Let the wave be $E_x = H_y = F\cos p(t-z/c)$. The number excited in the interval dt will be $NA_n(-Fp\sin pt)dt$. Consider the secondary wave crossing the point x, y, z at the time $t + r/c$. This is due to all the atoms which were excited before the time t. The number excited in the interval ds at a time $t-s$ is $NA_n(-Fp\sin p(t-s))ds$ and each of these will at the time t be giving a wave typified by $f = \alpha_n e^{-\lambda_n s}\cos k_n s$. So that total effect will be a wave which at the time $t + r/c$ at x, y, z has $x-$component

$$E_x = -NA_n Fp\left(-\frac{r^2-x^2}{r^3}\right)\int_0^\infty \sin p(t-s)ds\,.\,\alpha_n e^{-\lambda_n s}\cos k_n s \quad (2)$$

$$= NA_n\alpha_n\frac{r^2-x^2}{r^3}F\frac{p^2}{k^2-p^2}\cos pt$$

provided that λ_n is taken as small. The averaging has entirely blotted out the frequency of the atoms and left only that of the incident wave. Now on the classical theory, if there is a group of N_n electrons which have a natural frequency of vibration $k_n/2\pi$, the wave they scatter is given by

$$E_x = N_n\frac{e^2}{mc^2}\frac{r^2-x^2}{r^3}F\frac{p^2}{k^2-p^2}\cos pt.$$

So if we identify $N_n e^2/mc^2$ with $NA_n\alpha_n$ the expressions are the same. But the only difference between the phenomenon of scattering and of the refractive index lies in the matter of allowing for the mutual influence of the atoms, an influence exerted by the waves they send out and therefore the same on both theories. So we may at once say that from our result will follow the dispersion formula of Lorentz

$$\frac{3(\mu^2-1)}{\mu^2+2} = \Sigma_n\frac{4\pi NA_n\alpha_n c^2}{k_n{}^2-p^2}.$$

From the linear way in which the chance of excitation depends on the incident force, it follows that the average effects of superposed waves is additive; in other words the atoms act as Fourier analysers, sort out the harmonic components of an arbitrary incident wave and refract each component in the proper degree. In all cases the characteristics frequency with which the waves are really emitted will entirely disappear by averaging.

It will be necessary to consider the balance of energy which is nearly but not quite exact, but the present simple equations are not suited for this; they fail to give the balance even in the classical case, and there it must occur. This question is better treated in connection with absorption. The problem is complicated by the fact that the excited wave may possibly have a phase differing slightly (it may only be slightly) from that of a cosine. I have assumed the form of the damping factor as $e^{-\lambda_n t}$ only for convenience; all that is necessary is that the infinite end should be unimportant. An alternative is to suppose that the wave is undamped but that there is a chance $\lambda_n dt$ in every element of time dt that it should stop. We have only discussed waves polarised along the x-axis and have supposed that the excited waves have this axis as pole; for the general case the formulation must be somewhat changed, but it would take too long to state and prove the modification here. The essential points of the theory are not altered, and it also appears that there should be no particular difficulty in fitting double refraction and rotatory dispersion into our scheme.

A theory of dispersion is not of course complete without including selective absorption. If λ_n is retained in the integration of (2) the result is an expression practically the same as that given in the classical theory when a damping factor is included. Observe that on the present theory when the forced period approaches the natural, there is no increase either in the number of atoms excited or in the strength of the waves they send out. The whole change is due to the greater efficiency with which they reinforce the primary beam. Our theory gives no explanation of the mechanism of conversion of radiant energy into atomic heat, any more than does the classical theory with its damping factor. The conversion is probably better studied by the consideration of other cases of absorption, such as metallic reflexion, and our method of argument, applied to this last, should certainly give interesting results. We shall have to find what emission of spherical waves will diminish the aethereal energy when superposed on the incident wave. Thus a wave like that for dispersion would do for metallic reflexion, if the phase is suitably altered, or possibly we may suppose that the wave is again in the form of a cosine, but that the chance of excitation is now proportional to E_x instead of to $\partial E_x/\partial t$. It seems likely that a study of the optical constants of metals would throw light on this question. Afterwards it would be necessary to examine the

balance of energy between aether and matter and this might help in understanding the mechanism of the process.

We may now review how these speculations will modify the accepted theory. As we have made no assumptions as to what goes on inside the atom, we can take over the whole of the dynamics of stationary states. We suppose that an atom is usually in its lowest quantum state. The motions of the electrons will sometimes lead to a favourable configuration and when this occurs in the presence of a changing electric force there is a chance that the atom may be jerked into a condition in some way associated with one of its higher quantised states. It at once starts radiating with a frequency corresponding to the return from that state to the lowest. Dispersion throws no light on the amplitude of the wave, for in the formula it always occurs multiplied by the probability factor A_n. It is rather tempting to suppose that it actually goes into the higher quantised state, and then gives a wave of such amplitude and length that, but for the interference with the incident light, it would emit energy $hk_n/2\pi$. If this is so we may perhaps extend our theory to cover pure emission; for though we have not postulated any precise relationship between electric force and electrons, it seems inevitable that there should be a rapidly changing electric force near a moving electron, and this force would have a chance of jerking the atom into its higher state. On the other hand difficulties are raised in other directions. For the rediation must be immediate and therefore the state would not really be stationary at all, and the accepted theory of specific heats requires that a molecule should be able to remain in its higher states. In any case there is a clear contradiction to the principle of energy, but the phases of the outgoing waves are so adjusted that for cases of pure scattering or refraction on the average as much energy goes out as comes in.

There are many other points that will require attention. In the first place the refractive index is closely related to the dielectric constant. Now though it is quite proper to treat the dielectric constant as a limiting case of refraction yet it can be regarded electrostatically and it will be necessary to see the physical meaning of this aspect. Again it is possible to count the electrons in the atoms by X-ray reflexion, and it follows that there must be a relation between the e^2/mc^2 of the classical theory and our $A_n\alpha_n$. In this connection I owe to Prof. P. S. Epstein the suggestion that the theory may explain the defect observed in the scattering of hard γ-rays below that predicted. Here the wave length of the incident light is much shorter than the distances between the electrons and the incoherent waves cannot recombine in the way they do under the classical theory. Lastly, it will be necessary to reëxamine the deduction of the formula for black radiation, for all present proofs are founded on theorems following out of the conservation of energy.

In view of the great number of problems that are suggested and the probability that it will take a considerable time to deal with them, it appeared to me that it might be of interest to publish this preliminary account of a very incomplete theory.

* The difficulty is that the standard theory indicates a dispersion formula involving the frequency of the electron's motion in the atom, which is quite different from its absorption frequency.

CIRCUS MOVEMENTS OF LIMULUS

By William H. Cole

Lake Forest College

Communicated, December 4, 1922

By the symmetric and asymmetric stimulation by light of Limuli (from 20 to 60 mm. in diameter) it has been demonstrated that the animals are positively phototropic and that they show positive circus movements. It has further been found that the diameter of the circles turned as the result of asymmetric stimulation is inversely proportional to the light intensity. Because of the importance of circus movements in the analysis of orientation to light, and because there has recently appeared in these pages by Professor Mast a new explanation of orientation in insects, opposed to the tropism theory, the experiments leading to the above conclusions are briefly reported at this time. A fuller account will appear later in another journal.

In all the experiments on circus movements with Limuli the conditions were such that the animals were subjected to diffuse and non-directive illumination, the only conditions under which circus movements should ever be investigated. This was accomplished by arranging several Mazda lamps around the outside of a cylindrical glass dish lined on the inside with tissue paper. Three intensities of light were obtained by placing the lamps at different distances from the center of the dish, viz., 150, 300 and 450 mm., and although the intensities were not accurately determined, they were always in the same proportion, and were approximately 8000, 2000 and 900 candle meters. The conditions of asymmetry in the photoreceptors were produced either by removing or by blackening the two median eyes and one lateral eye, leaving the other lateral eye functional. One hundred thirty-five experiments on thirty-eight animals furnished the data presented here. The paths of the animals were transferred to record sheets, and were afterward measured as to length and number of degrees turned per centimeter. From table I it is seen that the number of degrees

TABLE 1

THE NUMBER OF DEGREES TURNED PER CENTIMETER UNDER THREE INTENSITIES OF LIGHT*

Candle Meters	Length of Path	Degrees per Centimeter
8000	374	6.73±0.18
2000	362	5.23±0.17
900	339	4.78±0.17

* Each number in the last two columns represents the mean of 45 determinations.

turned per centimeter varies directly with the light intensity, being 6.7 for 8000 cm.; 5.23 for 2000 cm., and 4.78 for 900 cm. In other words, the circles turned are larger under low intensity than they are under high intensity. To state the differences in more concrete form, it may be said that the diameters of the three circles turned are in the proportion of 100 to 128 to 142 for the corresponding intensities of 8000, 2000 and 900 candle meters.

The experimental animals were selected because of the regularity of their phototropic reactions. It was found that about 25% of normal freshly collected animals are very irregular in their reactions, some being indifferent to light, some being at first definitely positive and soon thereafter negative, and still others showing a mixture of positive and negative reactions. A higher percentage of irregularity is found in animals which have been in the laboratory several days. In spite of this irregularity in the phototropic reactions of Limulus, which may be due to several factors, such as the ease with which some individuals are frightened by handling, the state of nutrition, and unknown factors due to previous stimuli, yet it is clear that the animals taken as a whole are fundamentally positive to light, and that positive circus movements appear as a result of a symmetric stimulation. Many factors were found which modified or obliterated this fundamental phototropic reaction. Such behavior illustrates very well the fact that a primitive reaction of an animal may be profoundly modified by other reactions occurring simultaneously.

This is not at all a new idea, since it has long been known that there is an inclined place of animal behavior, beginning with the lowest forms whose reactions to stimuli, such as light, are machine-like in character, and ending with man whose reactions to the same stimuli are almost always modified or suppressed by reason, learning, etc. It is not surprising, therefore, to find that Limulus, the anatomy of whose nervous system has led to the belief that the animal corresponds closely to the hypothetical ancestors of the vertebrates, shows a modified tropistic reaction many times in respect to light.

The inverse relation between the light intensity and the diameter of the circle turned as a result of asymmetric stimulation contributes further evidence to Loeb's theory of heliotropism formulated in 1888. It should

be pointed out that although several investigators previous to that time had proposed theories of orientation somewhat like the tropism theory of Loeb, none of them embraced all the facts, and none of them were founded upon such conclusive evidence as Loeb's. It is absurd to claim that such early workers as Ray and de Candolle had the same idea as is expressed in the tropism theory concerning the photochemical effect of light upon the eye, since the true science of photochemistry was not even begun until about the middle of the nineteenth century.

When circus movements are performed by an animal in diffuse non-directive light, the strength of the photic stimulus remains practically constant throughout a single revolution, since the rays of light enter the eye at all angles with the same intensity. This fact would vitiate any hypothesis based upon localized retinal stimulation to explain circus movements in animals with eyes. The phototropism of animals without eyes cannot of course be explained by such a hypothesis, but is subject to explanation by the tropism theory. The evidence from Limulus as well as from other forms is preponderantly in favor of Loeb's theory, and the few real exceptions so far reported are undoubtedly due to the masking of phototropism by other more complex reactions occurring simultaneously.